# A Note on Non-Centered and Log-Concave Extensions of the Gaussian Correlation Inequality


**Abstract**

Royen proved the Gaussian correlation inequality for centered Gaussian measures and origin-symmetric convex sets. More recently, Nakamura and Tsuji extended the inequality to convex sets with the same Gaussian barycenter, hence in particular to all *centered* convex sets, i.e. convex sets that are symmetric about the origin ($K = -K$). In this note we record that two literal further extensions fail in the strongest possible sense. First, the naive analogue for arbitrary non-centered Gaussian measures and symmetric convex sets is false already in $\mathbb{R}^2$, even for positive-definite covariance matrices and for strips. Second, if $c(n)$ denotes the optimal constant such that

$$\mu(A \cap B) \geq c(n)\, \mu(A)\mu(B)$$

for every log-concave probability measure $\mu$ on $\mathbb{R}^n$ and all symmetric convex Borel sets $A, B \subset \mathbb{R}^n$, then

$$c(1) = 1, \qquad c(n) = 0 \quad (n \geq 2).$$

The proof is elementary and geometric.


## 1 Introduction

We call a convex set $K \subset \mathbb{R}^n$ *centered* (or *origin-symmetric*) if $K = -K$; this is distinct from requiring that $K$ contain the origin or that $K$ have Gaussian barycenter 0.

The Gaussian correlation inequality (GCI) states that for every centered Gaussian probability measure $\gamma$ on $\mathbb{R}^n$ and all centered convex Borel sets $K, L \subset \mathbb{R}^n$,

$$\gamma(K \cap L) \geq \gamma(K)\gamma(L). \tag{1}$$

The inequality was proved in dimension 2 by Pitt [1977]; important partial cases were later obtained by Schechtman et al. [1998], Hargé [1999], and Cordero-Erausquin [2002]. Royen [2014] proved (1) in full generality, and alternative proofs and expositions were subsequently given by Latała and Matlak [2017] and by Milman [2025]. Very recently, Nakamura and Tsuji [2025] established a strictly stronger theorem: (1) holds for convex sets with the same Gaussian barycenter, hence in particular for all centered convex sets.

Several natural questions arise once one attempts to remove the centering or Gaussian assumptions; two such questions are posed as open problems on SolveAll.org [SolveAll.org, 2026]. The purpose of this note is to show that two literal extensions fail. The first asks whether (1) remains valid for an arbitrary Gaussian measure $\mathcal{N}(m, \Sigma)$ and origin-symmetric convex sets. The second asks for the optimal dimension-dependent constant $c(n)$ in the class of all log-concave probability measures. Our main result shows that the former statement is false in every dimension $n \geq 2$, and that the latter problem has the trivial answer $c(1) = 1$ and $c(n) = 0$ for every $n \geq 2$.

For $n \geq 1$, let $c(n)$ be the supremum of all $c \in [0, 1]$ such that

$$\mu(A \cap B) \geq c\, \mu(A)\mu(B)$$

for every log-concave probability measure $\mu$ on $\mathbb{R}^n$ and all symmetric convex Borel sets $A, B \subset \mathbb{R}^n$. Since $0 \leq \mu(A \cap B)$, the quantity $c(n)$ is well defined.

**Theorem 1.**

1. *For every integer $n \geq 2$ there exist a vector $m \in \mathbb{R}^n$, a positive-definite covariance matrix $\Sigma$, and symmetric convex Borel sets*
$$A, B \subset \mathbb{R}^n$$
*such that*
$$\mathcal{N}(m, \Sigma)(A \cap B) < \mathcal{N}(m, \Sigma)(A)\, \mathcal{N}(m, \Sigma)(B).$$
2. *The optimal constants $c(n)$ satisfy*
$$c(1) = 1, \qquad c(n) = 0 \quad (n \geq 2).$$

**Remark 1.** *The geometry behind the counterexample is simple. The Gaussian measure will be concentrated near the affine line $y = 1$ in $\mathbb{R}^2$. Along that line the strip $\{|x| \leq 1/4\}$ cuts out the interval $[-1/4, 1/4]$ in the $x$-coordinate, while the strip $\{|x-y| \leq 1/4\}$ cuts out the interval $[3/4, 5/4]$. Thus each strip has positive mass, but their overlap near where the Gaussian mass lives is negligible.*

## 2 A Two-Dimensional Counterexample

Let $\Phi$ denote the standard normal distribution function. For $\varepsilon > 0$ define a Gaussian probability measure on $\mathbb{R}^2$ by

$$\mu_\varepsilon := \mathcal{N}\left( \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \right).$$

Its covariance matrix is positive definite.

Consider the two strips

$$A := \{(x, y) \in \mathbb{R}^2 : \ |x| \leq \tfrac{1}{4}\}, \qquad B := \{(x, y) \in \mathbb{R}^2 : \ |x - y| \leq \tfrac{1}{4}\}. \tag{2}$$

Both sets are closed, convex, and symmetric with respect to the origin.

**Proposition 2.** *For all sufficiently small $\varepsilon > 0$,*

$$\mu_\varepsilon(A \cap B) < \mu_\varepsilon(A)\mu_\varepsilon(B).$$

*In particular, the non-centered Gaussian extension of the Gaussian correlation inequality fails already in dimension 2.*

*Proof.* Let $Z, W$ be independent standard Gaussian random variables. Then

$$\mu_\varepsilon = \mathcal{L}(Z, 1 + \varepsilon W).$$

Since $A$ only constrains the first coordinate,

$$\mu_\varepsilon(A) = \mathbb{P}(|Z| \leq \tfrac{1}{4}) = 2\Phi(\tfrac{1}{4}) - 1 > 0. \tag{3}$$

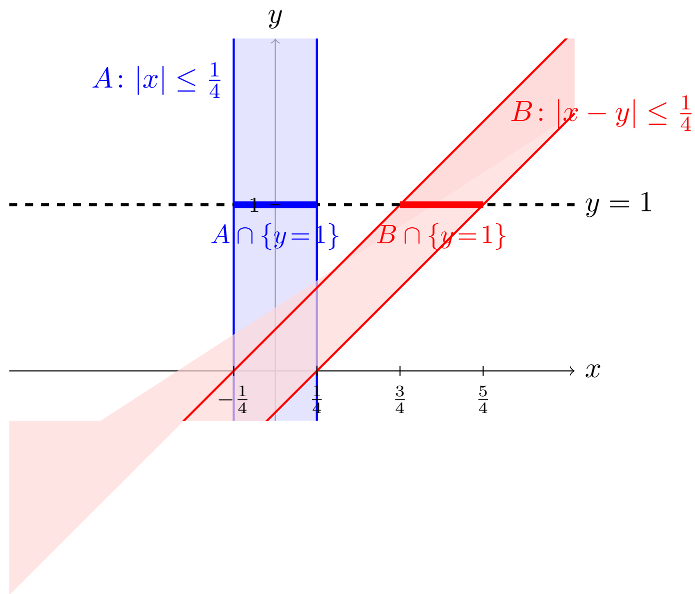


Figure 1: The counterexample geometry in $\mathbb{R}^2$. The vertical strip $A = \{|x| \le 1/4\}$ (blue) and the diagonal strip $B = \{|x-y| \le 1/4\}$ (red) are both centered convex sets. The dashed line $y = 1$ indicates where the Gaussian measure $\mu_\varepsilon$ concentrates as $\varepsilon \downarrow 0$. Along $y = 1$, the strip $A$ occupies $x \in [-\frac{1}{4}, \frac{1}{4}]$ and $B$ occupies $x \in [\frac{3}{4}, \frac{5}{4}]$; these intervals are disjoint, so $A \cap B$ has negligible mass near $y = 1$.

Next,

$$Z - (1 + \varepsilon W) \sim \mathcal{N}(-1, 1 + \varepsilon^2),$$

so

$$\mu_\varepsilon(B) = \mathbb{P}(|Z - (1 + \varepsilon W)| \le \tfrac{1}{4}) = \Phi\left(\frac{5}{4\sqrt{1+\varepsilon^2}}\right) - \Phi\left(\frac{3}{4\sqrt{1+\varepsilon^2}}\right). \tag{4}$$

Hence

$$\mu_\varepsilon(A)\mu_\varepsilon(B) \longrightarrow \big(2\Phi(\tfrac{1}{4}) - 1\big)\big(\Phi(\tfrac{5}{4}) - \Phi(\tfrac{3}{4})\big) > 0 \qquad (\varepsilon \downarrow 0). \tag{5}$$

On the other hand, if $(x, y) \in A \cap B$, then $|x| \le 1/4$ and $|x - y| \le 1/4$, and therefore

$$|y| = |x - (x - y)| \le |x| + |x - y| \le \tfrac{1}{2}.$$

Thus

$$A \cap B \subset \{(x, y) \in \mathbb{R}^2 : \ |y| \le \tfrac{1}{2}\},$$

and so

$$\mu_\varepsilon(A \cap B) \le \mathbb{P}(|1 + \varepsilon W| \le \tfrac{1}{2}) \le \mathbb{P}(1 + \varepsilon W \le \tfrac{1}{2}) = \Phi\left(-\frac{1}{2\varepsilon}\right) \longrightarrow 0 \qquad (\varepsilon \downarrow 0). \tag{6}$$

Combining (5) and (6), we obtain

$$\mu_\varepsilon(A \cap B) < \mu_\varepsilon(A)\mu_\varepsilon(B)$$

for all sufficiently small $\varepsilon > 0$. $\square$

**Corollary 3.** *For every integer $n \geq 2$ there exist $m \in \mathbb{R}^n$, a positive-definite covariance matrix $\Sigma$, and symmetric convex Borel sets $A_n, B_n \subset \mathbb{R}^n$ such that*

$$\mathcal{N}(m, \Sigma)(A_n \cap B_n) < \mathcal{N}(m, \Sigma)(A_n)\,\mathcal{N}(m, \Sigma)(B_n).$$

*Proof.* For $n = 2$ this is Proposition 2. Fix $n \geq 3$, let $\gamma_{n-2}$ denote the standard Gaussian measure on $\mathbb{R}^{n-2}$, and define

$$\nu_\varepsilon := \mu_\varepsilon \otimes \gamma_{n-2} = \mathcal{N}\big((0,1,0,\dots,0), \operatorname{diag}(1, \varepsilon^2, 1, \dots, 1)\big)\,.$$

Set

$$A_n := A \times \mathbb{R}^{n-2}, \qquad B_n := B \times \mathbb{R}^{n-2},$$

where $A, B \subset \mathbb{R}^2$ are the strips from (2). Then $A_n$ and $B_n$ are symmetric convex Borel sets, and Fubini's theorem gives

$$\nu_\varepsilon(A_n) = \mu_\varepsilon(A), \qquad \nu_\varepsilon(B_n) = \mu_\varepsilon(B), \qquad \nu_\varepsilon(A_n \cap B_n) = \mu_\varepsilon(A \cap B).$$

Hence Proposition 2 implies the desired strict inequality for all sufficiently small $\varepsilon > 0$. $\square$

## 3 The Log-Concave Constant

We next determine the optimal constants $c(n)$.

**Proposition 4.** *One has $c(1) = 1$.*

*Proof.* Let $\mu$ be an arbitrary probability measure on $\mathbb{R}$. Every centered convex Borel set in $\mathbb{R}$ is a centered interval: one of $\emptyset$, $[-a, a]$, $(-a, a)$, $\{0\}$, or $\mathbb{R}$ for some $a \in (0, \infty)$. In particular, any two such sets are nested (one is contained in the other up to a set of measure zero for any non-atomic measure). Since a probability measure can have at most countably many atoms, the boundary points $\pm a$ carry zero mass for all but countably many values of $a$, so for the purpose of computing $\mu$-measures we may write $A = [-a, a]$ and $B = [-b, b]$ with $a, b \in [0, \infty]$.

These sets are nested. Suppose, without loss of generality, that $a \leq b$. Then $A \cap B = A$, and therefore

$$\mu(A \cap B) = \mu(A) \geq \mu(A)\mu(B),$$

since $0 \leq \mu(B) \leq 1$. Thus the inequality with constant 1 holds for every probability measure on $\mathbb{R}$, so in particular for every log-concave probability measure. Because by definition $c(1) \leq 1$, we conclude that $c(1) = 1$. $\square$

**Proposition 5.** *For every integer $n \geq 2$, one has $c(n) = 0$.*

*Proof.* We first treat the case $n = 2$. Each measure $\mu_\varepsilon$ is Gaussian, hence log-concave. By (5) and (6),

$$\frac{\mu_\varepsilon(A \cap B)}{\mu_\varepsilon(A)\mu_\varepsilon(B)} \longrightarrow 0 \qquad (\varepsilon \downarrow 0).$$

Therefore, for every $c > 0$, one can choose $\varepsilon > 0$ small enough so that

$$\mu_\varepsilon(A \cap B) < c\,\mu_\varepsilon(A)\mu_\varepsilon(B).$$

No positive constant works in dimension 2, and since the constant 0 is trivially admissible, it follows that $c(2) = 0$.

Now fix $n \geq 3$. Consider the product Gaussian measures $\nu_\varepsilon = \mu_\varepsilon \otimes \gamma_{n-2}$ and the sets $A_n, B_n$ from the proof of Corollary 3. Then $\nu_\varepsilon$ is log-concave and

$$\frac{\nu_\varepsilon(A_n \cap B_n)}{\nu_\varepsilon(A_n)\nu_\varepsilon(B_n)} = \frac{\mu_\varepsilon(A \cap B)}{\mu_\varepsilon(A)\mu_\varepsilon(B)} \longrightarrow 0 \qquad (\varepsilon \downarrow 0).$$

Thus no positive constant works in dimension $n$, and again the trivial constant 0 is admissible. Hence $c(n) = 0$ for every $n \geq 3$. □

We have now proved Theorem 1. The argument shows that any meaningful non-centered extension of the Gaussian correlation inequality must impose an additional normalization, such as the common-barycenter condition introduced by Szarek and Werner [1999] and recently resolved in full by Nakamura and Tsuji [2025].

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